I. Solution of a Problem in Skeleton Division.

By D. R. Curtiss, Northwestern University.

In the June–July number of this Monthly (*1921*, 278) mention was made of a problem in skeleton division:

"The following calculation represents the development of an irreducible fraction to a repeating decimal:

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xxxxxx)xxxxxxx(xx.xxxxxxxxxx
       xxxxxx
       ------
       xxxxxxx
        xxxxxx
       -------
        xxxxxxx
        xxxxxxx
        -------
         xxxxxxx
          xxxxxx
         -------
          xxxxxxx
          xxxxxxx
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           xxxxxxx
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            xxxxxxx
            xxxxxxx
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             xxxxxxx
             xxxxxxx
             -------
                  xxxxxx
                  xxxxxx
                  ------
                  xxxxxx
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Crosses indicate digits—all of which are possible. The repeating digits are shown by a line above the crosses. What irreducible fraction has been developed?"

The divisor is 667334, the dividend 7752341, the quotient $11.6\overline{168830001}\cdots$, the repeating digits being the nine under the dash.

First, as to the divisor:—The quotient, as an unending decimal, can be expressed as an infinite series

$$Q = \frac{m}{10} + \frac{n}{10^{10}} + \frac{n}{10^{19}} + \cdots,$$

where m and n are whole numbers. By the formula for the sum of a geometric series we have

$$Q = \frac{m}{10} + \frac{n}{10^{10}} \cdot \frac{1}{1 - 10^{-9}} = \frac{M}{10(10^9 - 1)},$$

where M is a whole number. Then, since dividend and divisor are relative primes by hypothesis, the divisor is a whole number which is a factor of $10(10^9 - 1)$. The prime factors of this number are 2, 3^4, 5, 37 and 333667. Since our divisor has six digits, it must be either 333667 or 667334. The former of these, however, is not the divisor, because the first three digits of line 7, and therefore of line 19,

would have to be 100 if the divisor were 333667, and the last indicated digit of the quotient would be 1 or 2. Multiply either into 333667 and subtract the result from a number of six digits whose last four are zeros, and the remainder cannot be a number beginning with 100. Thus 667334 is the only possible divisor.

We now show that the number in line 17 is 780000. The last indicated digit of the quotient must be 1, or there would be seven digits in line 18, and 1 is also the fourth digit of the quotient. This means that line 7 and line 19 begin with 10 or 11 or 12 or 13. The number in line 17, which is the sum of those in lines 18 and 19 (the former is the divisor itself), must then begin with 77 or 78 or 79 or 80. But the numbers in lines 15 and 16 are both even (the former terminating in a 0, and the divisor being even), hence their difference is even, and we have only 78 or 80 as possible first two digits in line 17. We dispose of 80 by noting that the numbers in lines 13 and 14 are even, so line 15 has an even number and a zero as its last two digits; to begin line 17 with 80 line 16 must end with a zero so that the eighth digit of the quotient would be 5, and line 16 ends with 70. Subtract from line 15, which ends with an even digit followed by a zero and the remainder cannot be 80.

We now know lines 17 and 18, and in consequence 19 and 7, the latter being 1126660. As line 5 ends with a zero, line 6 ends with a 4, and the third digit of the quotient is 6, since line 6 has seven digits. Line 5 is therefore 4116670, the sum of 112666 and 4004004. The first and second digits of the quotient are each 1, since lines 2 and 4 have but six digits apiece. These lines are, then, 667334, and a couple of additions brings us back to the dividend 7752341.

Note that the two decisive points are the number of factors in the repetend and the small remainder at line 17. If one wishes to amuse himself by constructing similar problems he would best pick this small remainder first, put it near the end of his operations, and build up the problem from it.

Remarks by A. A. Bennett, University of Texas.

This puzzle in skeleton division appears at first glance to be unadapted to any direct attack and to be probably capable of many solutions. A very brief examination is sufficient, however, to identify the division as that indicated by 7752341/667334. It may first be noticed that the continued decimal, which may be generated by $\times\times\times\times\times 000\times/999,999,999$, and therefore by $\times\times\times\times\times\times/$ $999,999,999$, is also generated by a fraction of the form $\times\times/\times\times\times\times\times\times$, as seen by the next to the last remainder. In this last fraction, if the common factors of numerator and denominator be removed, one has in the denominator a factor of 999,999,999, and this factor is at least a four-digit number. The factorization of 999,999,999 gives $3^4 \times 37 \times 333,667$, the last factor being a prime. A brief examination of 333,667 shows that this cannot itself be the divisor. It remains to test $2 \times 333,667$, any higher multiples being necessarily of seven figures. Now every six-figure multiple of 667,334 must be this number itself. Using this fact the steps may be rapidly retraced and the gaps filled.

The only alternative would be for the fraction $\times\times/\times\times\times\times\times\times$ to be of the

form $ab/(2997a) = \times\times\times\times\times\times/999{,}999{,}999$. Then $b \times 333{,}667 = \times\times\times\times\times\times$. Hence $b = 1$, or 2 and $\times\times\times\times\times\times = 333{,}667$ or $667{,}334$. Both suggested possibilities are readily tested and excluded, by virtue of the fact that the last remainder has six digits prior to carrying down the next zero. Thus for $b = 1$, we have a in the form $\times\times$ and $3a \times 2997$ or $8991a$ as the last quantity to subtract. In other words $(10000 - 8991)a$ is the remainder, or $1009a$, where a is a two-digit number. Even 99 is too small for a, so that $b = 1$ leads to a contradiction as before. The solution then is unique.

II. What is a Calculus?

By J. P. Ballantine, University of Michigan.

There is much room for speculation as to the possible ways in which the calculus can be generalized. With a hope of stimulating other answers, rather than any presumption of closing the question, I suggest the following answer.

Consider the functions (1), which we may call the elementary functions of the Newtonian calculus, or briefly the Newtonian elementary functions:

$$0, \quad 1, \quad \frac{x}{1!}, \quad \frac{x^2}{2!}, \quad \frac{x^3}{3!} \cdots. \tag{1}$$

Notice that the derivative of each of the set is its predecessor. In fact, the process of differentiation may be regarded superficially as the process of replacing any of the above functions as they appear in a linear combination by its predecessor.

Thus, if $F(x)$ is given by the equation:

$$F(x) = a + \frac{bx}{1!} + \frac{cx^2}{2!} + \frac{dx^3}{3!} + \cdots, \tag{2}$$

we may infer (under certain conditions of convergence) that the derivative $F'(x)$ is given by the equation

$$F'(x) = b + \frac{cx}{1!} + \frac{dx^2}{2!} + \cdots. \tag{2'}$$

The object of this interpretation of the process of forming a derivative is that it admits readily of a generalization. One can substitute in place of the functions (1) another set of elementary functions, and if as a set they satisfy certain conditions, one can build up a new calculus based on them. The process of differentiation is not altered, but the geometrical interpretation is. Take for instance the set of functions,

$$0, \quad 1, \quad \frac{x}{1}, \quad \frac{x(x-1)}{1\cdot 2}, \quad \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}. \tag{1'}$$

The calculus based on these is the well-known calculus of finite differences.